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Finite-Difference Methods for Calculating Steady Incompressible Flows in Three Dimensions

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A new stable numerical method is described for solving the Navier–Stokes equations for the steady motion of an incompressible fluid in three dimensions. The basic governing equations are expressed in terms of three equations for the velocity components together with three equations for the vorticity components. This gives six simultaneous coupled second-order partial differential equations to be solved. A finite-difference scheme with second-order accuracy is described in which the associated matrices are diagonally dominant. Numerical results are presented for the flow inside a cubical box due to the motion of one of its sides moving parallel to itself for Reynolds numbers up to 100. Several methods of approximation are considered and the effect of different discretizations of the boundary conditions is also investigated. The main method employed is stable for Reynolds numbers greater than 100 but a finer grid size would be required in order to obtain accurate results.

1. INTRODUCTION

There has been much interest in recent times in new finite-difference schemes for approximating partial differential equations which arise in the numerical solution of the Navier–Stokes equations governing the motion of a viscous fluid. A typical case arises in the approximation of the vorticity transport equation for the steady two-dimensional flow of an incompressible fluid. It is possible by the use of specialized techniques to obtain an approximating set of finite-difference equations which have coefficients involving exponentials. These equations have properties which suggest that they may be sometimes superior to the more standard methods of approximation in the associated computational procedures. The properties are similar to those of

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the upwind and downwind types of differencing schemes but the equations can be shown to approximate more accurately to the basic differential equation than these schemes.

The basis of the exponential type of approximation goes back to a paper by Allen and Southwell [1] which studied the steady two-dimensional flow past a circular cylinder. The method was examined in detail in a subsequent paper by Dennis [2] who also proposed an alternative method in this two-dimensional case. Later investigations of a similar nature have been given by Spalding [3], Dennis [4], and Roscoe [5]. An extension to three-dimensional flows has been made by Roscoe [6]. In the present paper the extension of the alternative method to Allen and Southwell's method of approximation given by Dennis [2] and Dennis and Hudson [7] is made to problems in three dimensions governed by the Navier–Stokes equations for the steady flow of incompressible fluids

Several different finite-difference formulations of the steady-state three-dimensional Navier–Stokes equations have previously been investigated but usually they reduce to one of two approaches. The set of partial differential equations can be solved in what are called the primitive variables, i.e., velocity components and pressure, and Chorin [8] and Williams [9] have investigated this approach. Difficulties arise with the pressure boundary conditions and also difficulty in satisfying continuity leads to instabilities. Aziz and Hellums [10] have formulated the Navier–Stokes equations in terms of the components of the vorticity and a vector potential. In either of these two approaches the steady-state equations are coupled and elliptic and must be solved iteratively. Mallinson and De Vahl Davis [11] developed a method of finding the steady-state solution by solving the unsteady equations using the method of false transients. An evaluation of upwind and central-difference approximations by means of a study of recirculating flows in two and three dimensions was given by De Vahl Davis and Mallinson [12]. Richardson and Cornish [13] have presented a method for solving three-dimensional incompressible flow problems including those governed by the Navier–Stokes equations. Finally, the most recent work on three-dimensional problems appears to be the study by Mallinson and De Vahl Davis [14] of natural convection in a rectangular cavity as a result of differential heating of the sides.

In the present paper a formulation of the three-dimensional Navier–Stokes equations for steady flow which bears some similarity to that of Aziz and Hellums [10] is used. The main difference is that three equations connecting the velocity and vorticity components are employed instead of a vector potential. These three equations together with the three expressing the transport of vorticity give six simultaneous second-order partial differential equations to be solved. This formulation was used by Cook [15] who employed central differences to approximate all partial derivatives. The present investigation uses the specialized finite-difference treatment described by Dennis [2], which is applied to the equations governing the transport of vorticity by a direct extension of the two-dimensional method. Unfortunately these finite-difference equations involve exponential functions and the associated matrices are not necessarily diagonally dominant (Varga [16]). Dennis and Hudson [7] showed how these two difficulties could be avoided and still maintain second-order accuracy.

This formulation is extended into three dimensions and used in the present paper. The three equations of Poisson type governing the velocity components are approximated by standard finite-difference methods. The usual no-slip boundary conditions for the velocity components are assumed on solid boundaries. Values of the vorticity components on solid boundaries are calculated, where necessary, by means of the equations which express these components in terms of derivatives of the velocity components. Two alternative approximations to these derivatives are investigated from the point of view of accuracy. Standard iterative procedures are used for solving the resulting finite-difference equations. A numerical example is considered in which the flow inside a cubical box due to the motion of one of its sides parallel to itself is computed. The results are compared with the results of the calculations of De Vahl Davis and Mallinson [12] and Cook [15] and the experimental results of Pan and Acrivos [17].

2. Basic Equations

The following formulation can be developed from the vector potential method of Aziz and Hellums [10] but will be given directly instead. The steady-state Navier–Stokes equations in three dimensions in the absence of any external forces may be written in the form

$$\nabla(\tfrac{1}{2}q^2) - \mathbf{q} \times (\nabla \times \mathbf{q}) = -\frac{1}{\rho}\nabla p + \nu\,\nabla^2\mathbf{q}, \tag{1}$$

where $\mathbf{q}$ is the velocity vector, ρ the density, p the pressure, and ν the kinematic viscosity. The continuity equation is given by

$$\nabla \cdot \mathbf{q} = 0. \tag{2}$$

If we take the curl of Eq. (1) to eliminate the pressure we obtain, in nondimensional component form,

$$R^{-1}\,\nabla^2\xi = u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y} + w\frac{\partial\xi}{\partial z} - \xi\frac{\partial u}{\partial x} - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z}, \tag{3a}$$

$$R^{-1}\,\nabla^2\eta = u\frac{\partial\eta}{\partial x} + v\frac{\partial\eta}{\partial y} + w\frac{\partial\eta}{\partial z} - \xi\frac{\partial v}{\partial x} - \eta\frac{\partial v}{\partial y} - \zeta\frac{\partial v}{\partial z}, \tag{3b}$$

$$R^{-1}\,\nabla^2\zeta = u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} + w\frac{\partial\zeta}{\partial z} - \xi\frac{\partial w}{\partial x} - \eta\frac{\partial w}{\partial y} - \zeta\frac{\partial w}{\partial z}, \tag{3c}$$

where (x, y, z) are the Cartesian coordinates, (u, v, w) the corresponding velocity components, and (ξ, η, ζ) the vorticity components which are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{4}$$

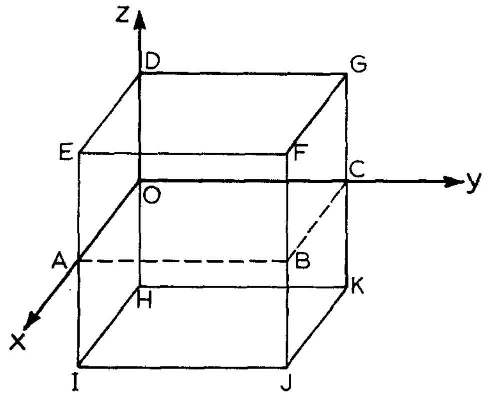

FIG. 1. The geometry.

All quantities are assumed nondimensional and $R = Ud/\nu$ is the Reynolds number based on a representative velocity U and length d. By use of the continuity equation (2), Eqs. (4) can be written

$$\nabla^2 u = \frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}, \qquad \nabla^2 v = \frac{\partial \zeta}{\partial x} - \frac{\partial \xi}{\partial z}, \qquad \nabla^2 w = \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x}. \tag{5}$$

It can be shown that satisfying Eqs. (3) and (5) implies that the equation of continuity (2) is automatically satisfied.

The three-dimensional square cavity DEFGHIJK of edge length d in which the solutions are to be obtained is shown in Fig. 1. The motion in the cavity is driven by the surface GFJK sliding with speed U in the positive x direction and all other surfaces of the cube are held at rest. The plane OABC ($z = 0$) is half way between the planes DEFG and HIJK and the flow is symmetrical about this plane. Thus the Navier–Stokes equations have to be solved only in the upper half of the box $z \geqslant 0$ and the boundary conditions are

$$\begin{aligned}
&\text{On } x = 0 \text{ and } x = 1\text{: } u = v = w = 0;\ \xi = 0,\ \eta = -\partial w/\partial x,\ \zeta = \partial v/\partial x.\\
&\text{On } y = 0\text{: } u = v = w = 0;\ \xi = \partial w/\partial y,\ \eta = 0,\ \zeta = -\partial u/\partial y.\\
&\text{On } y = 1\text{: } u = 1,\ v = w = 0;\ \xi = \partial w/\partial y,\ \eta = 0,\ \zeta = -\partial u/\partial y.\\
&\text{On } z = 0\text{: } \partial u/\partial z = \partial v/\partial z = w = 0;\ \xi = 0,\ \eta = 0,\ \partial\zeta/\partial z = 0.\\
&\text{On } z = \tfrac{1}{2}\text{: } u = v = w = 0;\ \xi = -\partial v/\partial z,\ \eta = \partial u/\partial z,\ \zeta = 0.
\end{aligned} \tag{6}$$

3. FINITE-DIFFERENCE EQUATIONS

The region of integration is covered by a cubical grid with elements whose sides are of length h and which are parallel to the x, y, and z directions. The present work is confined to taking equal grid sizes in all three directions, although this is obviously capable of generalization to unequal grid sizes in the three directions. In order to simplify the notation for subscripts, the scheme indicated in Fig. 2 is employed. In this a quantity at a typical point (x_0, y_0, z_0) is denoted by means of a subscript 0 and the subscripts 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 denote quantities at the points $(x_0 + h, y_0, z_0)$, $(x_0, y_0 + h, z_0)$, $(x_0 - h, y_0, z_0)$, $(x_0, y_0 - h, z_0)$, $(x_0, y_0, z_0 + h)$, $(x_0, y_0, z_0 - h)$, $(x_0 + 2h, y_0, z_0)$, $(x_0, y_0 + 2h, z_0)$, $(x_0 - 2h, y_0, z_0)$, $(x_0, y_0 - 2h, z_0)$, $(x_0, y_0, z_0 + 2h)$, and $(x_0, y_0, z_0 - 2h)$, respectively.

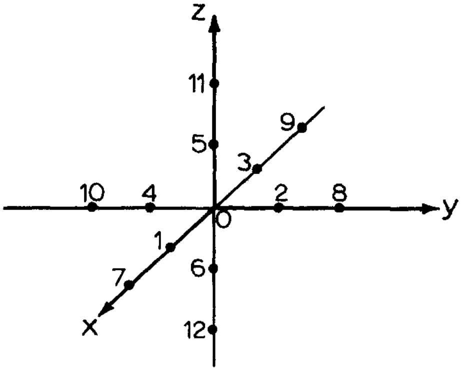

FIG. 2. Notation for the grid points.

Equations (5) and (3) can be written, using central-difference formulations to an accuracy which is of order h^2, in the forms

$$\begin{aligned} u_0 &= \frac{1}{6}\left[u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + \frac{h}{2}(\zeta_2 - \zeta_4 - \eta_5 + \eta_6)\right], \\ v_0 &= \frac{1}{6}\left[v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + \frac{h}{2}(\xi_5 - \xi_6 - \zeta_1 + \zeta_3)\right], \\ w_0 &= \frac{1}{6}\left[w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + \frac{h}{2}(\eta_1 - \eta_3 - \xi_2 + \xi_4)\right], \end{aligned} \tag{7}$$

$$\begin{aligned} \xi_0 = \frac{1}{6}\Big[&\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 \\ &- \frac{Rh}{2}\{u_0(\xi_1 - \xi_3) + v_0(\xi_2 - \xi_4) + w_0(\xi_5 - \xi_6) \\ &- \xi_0(u_1 - u_3) - \eta_0(u_2 - u_4) - \zeta_0(u_5 - u_6)\}\Big], \end{aligned} \tag{8a}$$

$$\begin{aligned} \eta_0 = \frac{1}{6}\Big[&\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6 \\ &- \frac{Rh}{2}\{u_0(\eta_1 - \eta_3) + v_0(\eta_2 - \eta_4) + w_0(\eta_5 - \eta_6) \\ &- \xi_0(v_1 - v_3) - \eta_0(v_2 - v_4) - \zeta_0(v_5 - v_6)\}\Big], \end{aligned} \tag{8b}$$

$$\zeta_0 = \frac{1}{6}\Big[\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6$$
$$- \frac{Rh}{2}\{u_0(\zeta_1 - \zeta_3) + v_0(\zeta_2 - \zeta_4) + w_0(\zeta_5 - \zeta_6)$$
$$- \xi_0(w_1 - w_3) - \eta_0(w_2 - w_4) - \zeta_0(w_5 - w_6)\}\Big]. \tag{8c}$$

Equations (7) determine u, v, and w and Eqs. (8) determine ξ, η, and ζ.

Although the matrices associated with Eqs. (7) are diagonally dominant those associated with the Eqs. (8) may not be and the solution of Eqs. (8) can only be expected to be obtained, at large values of the Reynolds number, if these equations are solved using an underrelaxation parameter, α say. Equations (7) are diagonally dominant and therefore can be solved using an overrelaxation parameter, β say. In order to overcome these difficulties Eq. (3) can be written using upwind and downwind differencing as described by Greenspan [18] and Gosman *et al.* [19] but with the result that the accuracy is reduced to order h. Alternatively, an extension of the method derived by Dennis [2] may be used, which will only be summarized briefly. Equation (3a) is written in the form of three equations

$$\frac{\partial^2 \xi}{\partial x^2} - Ru\frac{\partial \xi}{\partial x} = Q - A - B, \tag{9a}$$

$$\frac{\partial^2 \xi}{\partial y^2} - Rv\frac{\partial \xi}{\partial y} = A, \tag{9b}$$

$$\frac{\partial^2 \xi}{\partial z^2} - Rw\frac{\partial \xi}{\partial z} = B, \tag{9c}$$

where $Q = -R(\xi\, \partial u/\partial x + \eta\, \partial u/\partial y + \zeta\, \partial u/\partial z)$ and A and B are unknown functions of x, y, and z. Equation (9a) is solved along the grid segment $(x_0 - h, y_0, z_0)$ to $(x_0 + h, y_0, z_0)$ by first applying a transformation which removes the first-derivative term from the left side and then approximating the equation which results from this transformation by central differences at the point (x_0, y_0, z_0). After this approximation has been made, the new dependent variable of the transformation is replaced by the variable ξ. A similar procedure is applied to (9b) along the grid segment $(x_0, y_0 - h, z_0)$ to $(x_0, y_0 + h, z_0)$ and likewise to Eq. (9c) along the grid segment $(x_0, y_0, z_0 - h)$ to $(x_0, y_0, z_0 + h)$. All three of the approximating equations involve the unknown quantities A_0 and B_0, the local values of A and B at (x_0, y_0, z_0). Elimination of A_0 and B_0 gives the finite-difference equation

$$\xi_1 e^{r_1} + \xi_2 e^{s_2} + \xi_3 e^{r_3} + \xi_4 e^{s_4} + \xi_5 e^{t_5} + \xi_6 e^{t_6} - [6 + \tfrac{1}{4}R^2h^2(u_0{}^2 + v_0{}^2 + w_0{}^2)]\,\xi_0$$
$$= -\tfrac{1}{2}Rh[\xi_0(u_1 - u_3) + \eta_0(u_2 - u_4) + \zeta_0(u_5 - u_6)], \tag{10a}$$

where

$$r(x, y_0, z_0) = -\tfrac{1}{2}R \int_{x_0}^{x} u(\theta, x_0, y_0)\, d\theta,$$

$$s(x_0, y, z_0) = -\tfrac{1}{2}R \int_{y_0}^{y} v(x_0, \theta, z_0)\, d\theta,$$

$$t(x_0, y_0, z) = -\tfrac{1}{2}R \int_{z_0}^{z} w(x_0, y_0, \theta)\, d\theta.$$

Equations (3b) and (3c) can be put in finite-difference form in a similar way and the approximations which result are

$$\eta_1 e^{r_1} + \eta_2 e^{s_2} + \eta_3 e^{r_3} + \eta_4 e^{s_4} + \eta_5 e^{t_5} + \eta_6 e^{t_6} - [6 + \tfrac{1}{4}R^2h^2(u_0^2 + v_0^2 + w_0^2)]\,\eta_0$$
$$= -\tfrac{1}{2}Rh[\xi_0(v_1 - v_3) + \eta_0(v_2 - v_4) + \zeta_0(v_5 - v_6)], \tag{10b}$$

$$\zeta_1 e^{r_1} + \zeta_2 e^{s_2} + \zeta_3 e^{r_3} + \zeta_4 e^{s_4} + \zeta_5 e^{t_5} + \zeta_6 e^{t_6} - [6 + \tfrac{1}{4}R^2h^2(u_0^2 + v_0^2 + w_0^2)]\,\zeta_0$$
$$= -\tfrac{1}{2}Rh[\xi_0(w_1 - w_3) + \eta_0(w_2 - w_4) + \zeta_0(w_5 - w_6)]. \tag{10c}$$

The equations of type (10) are solved by iterative procedures for the three functions ξ, η, and ζ. During the course of the solution procedure the contributions to these equations from u, v, and w are temporarily held fixed at their current values. Also, in solving a given equation of type (10) it is assumed, for the purpose of the following discussion, that the right side is also held fixed at its current value. The associated matrix of a given equation is then considered to be that associated with the left side. This associated matrix is not necessarily diagonally dominant. Although the six exponential coefficients must necessarily be positive it cannot be shown that their sum will not exceed the magnitude of the coefficient with subscript zero, which is the required condition. In fact, for high Reynolds numbers some of the exponential coefficients may become large and the condition for diagonal dominance must almost certainly be violated. However, following Dennis and Hudson [7], by expanding the exponentials and retaining terms of order h^2 Eqs. (10) can be approximated by

$$\begin{aligned}
&(1 - \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\xi_1 + (1 - \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\xi_2\\
&\quad + (1 + \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\xi_3 + (1 + \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\xi_4\\
&\quad + (1 - \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\xi_5 + (1 + \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\xi_6\\
&\quad - [6 + \tfrac{1}{4}R^2h^2(u_0^2 + v_0^2 + w_0^2)]\,\xi_0\\
&= -\tfrac{1}{2}Rh[\xi_0(u_1 - u_3) + \eta_0(u_2 - u_4) + \zeta_0(u_5 - u_6)],
\end{aligned} \tag{11a}$$

$$\begin{aligned}
&(1 - \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\eta_1 + (1 - \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\eta_2\\
&\quad + (1 + \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\eta_3 + (1 + \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\eta_4\\
&\quad + (1 - \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\eta_5 + (1 + \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\eta_6\\
&\quad - [6 + \tfrac{1}{4}R^2h^2(u_0^2 + v_0^2 + w_0^2)]\,\eta_0\\
&= -\tfrac{1}{2}Rh[\xi_0(v_1 - v_3) + \eta_0(v_2 - v_4) + \zeta_0(v_5 - v_6)],
\end{aligned} \tag{11b}$$

$$\begin{aligned}
&(1 - \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\zeta_1 + (1 - \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\zeta_2 \\
&\quad + (1 + \tfrac{1}{2}Rhu_0 + \tfrac{1}{8}R^2h^2u_0^2)\,\zeta_3 + (1 + \tfrac{1}{2}Rhv_0 + \tfrac{1}{8}R^2h^2v_0^2)\,\zeta_4 \\
&\quad + (1 - \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\zeta_5 + (1 + \tfrac{1}{2}Rhw_0 + \tfrac{1}{8}R^2h^2w_0^2)\,\zeta_6 \\
&\quad - [6 + \tfrac{1}{4}R^2h^2(u_0^2 + v_0^2 + w_0^2)]\,\zeta_0 \\
&= -\tfrac{1}{2}Rh[\xi_0(w_1 - w_3) + \eta_0(w_2 - w_4) + \zeta_0(w_5 - w_6)]. \qquad (11c)
\end{aligned}$$

Since each coefficient of the terms with subscripts 1, 2, 3, 4, 5, and 6 is easily seen to be positive for all values of u_0, v_0, w_0, h, and R then the sum of these coefficients is equal in magnitude to the coefficient of the term with subscript zero and the associated matrices are diagonally dominant under all circumstances.

Diagonal dominance of the matrices associated with the left sides of the equations of type (11) is a desirable property from the practical point of view of obtaining numerical solutions. It must be noted, however, that Eqs. (11) have been derived by a procedure of power series expansion of exponentials in which it is assumed that Rh is reasonably small. Thus if Rh is large it may not be assumed that solutions obtained using (11) will necessarily be accurate even if they can be obtained more readily than by using the standard central-difference equations (8). In this sense Eqs. (11) are presented only as a possible formulation having second-order accuracy of a somewhat different character to the second-order accuracy of the system (8). The question of the accuracy of solutions can only effectively be considered by performing solutions for different grid sizes and comparing the results. A discussion of the computed results from this point of view is given later.

Equations (8), (10), and (11) represent three possible formulations of the differential equations (3), all of which are accurate to order h^2. The boundary conditions (6) must now be put in finite-difference form. Derivatives in the boundary conditions on the wall of the cube can be approximated by either forward or backward differences which have a truncation error of order h^2. Thus the boundary conditions (6) can be written

$$\begin{aligned}
&\text{On } x = 0, && u_0 = v_0 = w_0 = 0; \\
& && \xi_0 = 0,\ \eta_0 = -(4w_1 - w_7)/2h,\ \zeta_0 = (4v_1 - v_7)/2h. \\
&\text{On } x = 1, && u_0 = v_0 = w_0 = 0; \\
& && \xi_0 = 0,\ \eta_0 = (4w_3 - w_9)/2h,\ \zeta_0 = -(4v_3 - v_9)/2h. \\
&\text{On } y = 0, && u_0 = v_0 = w_0 = 0; \\
& && \xi_0 = (4w_2 - w_8)/2h,\ \eta_0 = 0,\ \zeta_0 = -(4u_2 - u_8)/2h. \qquad (12) \\
&\text{On } y = 1, && u_0 = 1,\ v_0 = w_0 = 0; \\
& && \xi_0 = -(4w_4 - w_{10})/2h,\ \eta_0 = 0,\ \zeta_0 = (4u_4 - u_{10} - 3)/2h. \\
&\text{On } z = 0, && u_0 = (4u_5 - u_{11})/3,\ v_0 = (4v_5 - v_{11})/3,\ w_0 = 0; \\
& && \xi_0 = 0,\ \eta_0 = 0,\ \zeta_0 = (4\zeta_5 - \zeta_{11})/3. \\
&\text{On } z = \tfrac{1}{2}, && u_0 = v_0 = w_0 = 0; \\
& && \xi_0 = (4v_6 - v_{12})/2h,\ \eta_0 = -(4u_6 - u_{12})/2h,\ \zeta_0 = 0.
\end{aligned}$$

An alternative form of the finite-difference representation of the boundary conditions may be used in which, rather than introducing the velocity components at a distance $2h$ from the solid boundaries, the values of the vorticity at a distance h from the solid boundary are used. Thus boundary conditions can also be approximated by

$$
\begin{array}{ll}
\text{On } x = 0, & u_0 = v_0 = w_0 = 0; \\
& \xi_0 = 0,\ \eta_0 = -2w_1/h - \eta_1,\ \zeta_0 = 2v_1/h - \zeta_1 . \\
\text{On } x = 1, & u_0 = v_0 = w_0 = 0; \\
& \xi_0 = 0,\ \eta_0 = 2w_3/h - \eta_3,\ \zeta_0 = -2v_3/h - \zeta_3 . \\
\text{On } y = 0, & u_0 = v_0 = w_0 = 0; \\
& \xi_0 = 2w_2/h - \xi_2,\ \eta_0 = 0,\ \zeta_0 = -2u_2/h - \zeta_2 . \\
\text{On } y = 1, & u_0 = 1,\ v_0 = w_0 = 0; \\
& \xi_0 = -2w_4/h - \xi_4,\ \eta_0 = 0,\ \zeta_0 = -2(1 - u_4)/h - \zeta_4 . \\
\text{On } z = 0, & u_0 = (4u_5 - u_{11})/3,\ v_0 = (4v_5 - v_{11})/3,\ w_0 = 0; \\
& \xi_0 = 0,\ \eta_0 = 0,\ \zeta_0 = (4\zeta_5 - \zeta_{11})/3. \\
\text{On } z = \tfrac{1}{2}, & u_0 = v_0 = w_0 = 0; \\
& \xi_0 = 2v_6/h - \xi_6,\ \eta_0 = -2u_6/h - \eta_6,\ \zeta_0 = 0.
\end{array}
\tag{13}
$$

In the iterative procedure it was sometimes found to be necessary to use a relaxation parameter λ when calculating the vorticity boundary values. Computations were performed using the finite-difference representation of the equations in the forms (7) and (8), (7) and (10), and (7) and (11). Also, computations were performed using the boundary conditions expressed in both of the finite-difference forms (12) and (13). All calculations were started from the initial assumption that $u = v = w = \xi = \zeta = \eta = 0$ everywhere except on $y = 1$, where $u = 1$. One complete iteration over all the internal grid points was performed using Eqs. (7) followed by one complete iteration over all the internal grid points using either Eqs. (8), (10), or (11). New boundary conditions were then calculated on all surfaces according to either Eqs. (12) or (13). This sequence is defined to be one iteration and was repeated until convergence. The criteria used for convergence were that at every internal grid point in the domain

$$| 1 - \xi_0^{(m)}/\xi_0^{(m+1)} | < \epsilon, \qquad | 1 - \eta_0^{(m)}/\eta_0^{(m+1)} | < \epsilon, \qquad | 1 - \zeta_0^{(m)}/\zeta_0^{(m+1)} | < \epsilon,$$

where ϵ is an assigned tolerance (usually taken to be 10^{-4}) and the superscripts denote the number of iterations. In each iteration the order in which the mesh points were swept was preserved.

4. Results

All the numerical computations were performed on the 1906A computer at Leeds University and results were obtained for $R = 0, 1, 10, 100, 400$ using various grid sizes $h = 0.1, 0.05, 0.04$. It was found that for Reynolds numbers up to 100 accurate and consistent results could be obtained using either formulations (8), (10), or (11) for the vorticity equations using a mesh size $h = 0.05$ provided that the boundary conditions were discretized in the form of Eqs. (13). Further, no underrelaxation was required and although the results could be obtained faster using overrelaxation, especially in the parameters α and β, all the results presented here are for the standard Gauss–Seidel procedure. If the boundary conditions are used as discretized in Eqs. (12) then a smaller grid size is required in order to obtain accurate results. This is illustrated for the case $R = 100$ in Fig. 3 which gives the u component of velocity on $z = 0$ at $y = 0.9$ as a function of x using the finite-difference formulation given by Eqs. (11) and the boundary condition given by Eqs. (12) and (13) with $h = 0.1$ and 0.05. Also shown are the two-dimensional results obtained by using the method described by Burggraf [20], where it is assumed that the box is infinite in the z direction. The results are graphically indistinguishable if the finite-difference formulations (8) and (10) are used rather than formulation (11). The use of a grid size $h = 0.04$ gives results which are in good agreement with those for $h = 0.05$ when using the boundary conditions in the form (13), but when using the boundary conditions (12) they move more closely toward the results obtained using $h = 0.05$ and Eqs. (13).

The results presented here are in good agreement with those obtained by Cook [15] who solved the finite-difference equations in the forms (7) and (8) along with the boundary conditions given by Eqs. (12). For example, Cook gave a value of $\zeta =$

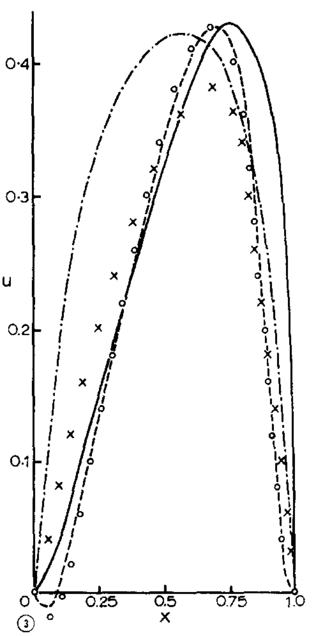

FIG. 3. The variation of u at $R = 100$, $z = 0$, and $y = 0.9$ as a function of x. ○, Burggraf; ×, $h = 0.1$ using (13); - - - - -, $h = 0.1$ using (12); ——, $h = 0.05$ using (12); – – –, $h = 0.05$ using (13).

-6.02 at $x = \frac{1}{2}$, $y = 1$, and $z = 0$ for the case $R = 1$ which was confirmed in the present calculations, whereas using Eqs. (7) and (11) along with the boundary conditions given by Eqs. (13) a value $\zeta = -5.88$ was obtained. At the corresponding point in the two-dimensional flow considered by Burggraf, Cook obtained a value $\zeta = -5.82$. The probable reason why the finite-difference formulation of the boundary conditions is more accurate in the form (13) rather than (12) is that the use of values at grid points which are at a distance 2h from a fixed boundary tend to smooth out the solution too much. Thus Eqs. (13) were used for the majority of the numerical results and only these results will be presented.

It is possible to calculate particle tracks for the motion and the results obtained are similar to those given by De Vahl Davis and Mallinson [12]. A direct comparison of these results is difficult but there is one property of the solutions which can readily be compared. De Vahl Davis and Mallinson have used a vector streamfunction $\boldsymbol{\psi}$ with components (ψ_x, ψ_y, ψ_z) in the three-dimensional problem. The component ψ_z in the z direction corresponds to the scalar streamfunction which may be employed to describe the two-dimensional motion in the xy plane which occurs when the box is of infinite length in the z direction. This component may be calculated for the present three-dimensional results in the plane $z = 0$ by suitable integrations over the xy plane and its maximum value in this plane compared with the corresponding calculated values of De Vahl Davis and Mallinson for both the three-dimensional and two-dimensional flows and with the results of Burggraf [20] in the two-dimensional case. The comparisons are shown for the case $R = 100$ in Table I for various grids in the xy plane at $z = 0$ and they clearly show that some experimentation with grid sizes is necessary in order to achieve accurate results. The present results were derived using Eqs. (11) and boundary conditions (13).

TABLE I

Comparison of the Maximum Value of ψ_z in the Plane $z = 0$ at $R = 100$[a]

Source	Grid	Maximum ψ_z
Ref. [12] (three-dimensional flow)	15×15	0.0867
Ref. [12] (two-dimensional flow)	15×15	0.0985
Ref. [20] (two-dimensional flow)	10×10	0.0784
	20×20	0.0955
	30×30	0.0999
	40×40	0.1015
	50×50	0.1022
Present (three-dimensional flow)	10×10	0.0769
	20×20	0.0945
	25×25	0.0956

[a] For two-dimensional flow ψ_z is the two-dimensional streamfunction in the plane of the solution.

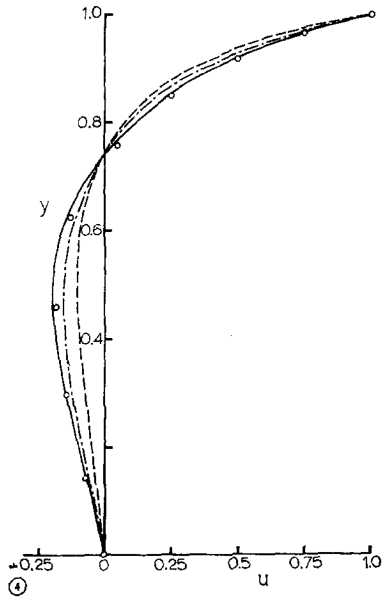

FIG. 4. The variation of u at $R = 100$ and $x = \frac{1}{2}$ as a function of y for various values of z. ○, Burggraf; ——, $z = 0$; - - - - -, $z = 0.3$; — — —, $z = 0.4$.

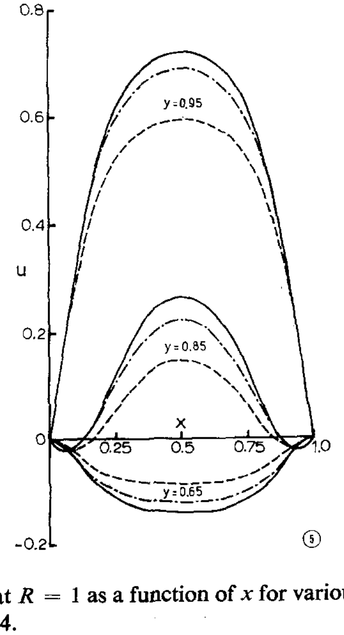

FIG. 5. The variation of u at $R = 1$ as a function of x for various values of y and z. ——, $z = 0$; - - - - -, $z = 0.3$; — — —, $z = 0.4$.

Because the results using the finite-difference formulations (8), (10), and (11) give good agreement up to values of $R = 100$ when using mesh sizes 0.05 and 0.04, the results presented below are for those obtained using Eqs. (11). Also the results for $h = 0.05$ and 0.04 cannot graphically be distinguished. Further, the computational times are very comparable, when using the simple Gauss–Seidel procedure, in all three formulations of the vorticity equations for Reynolds numbers up to 100.

Figure 4 shows the variation of u with y at $x = \frac{1}{2}$ and $z = 0$, 0.3, and 0.4 for $R = 100$. Also shown is the corresponding two-dimensional solution obtained using the method described by Burggraf. The results for $R = 0$ are very similar and are not shown. For both $R = 0$ and $R = 100$, the solution for u at $z = 0$ is very close to the Burggraf solution. Figures 5 and 6 show the variation of u as a function of x for $z = 0$, 0.3, and 0.4 and $y = 0.95$, 0.85, and 0.65. Figures 4–6 show the same kind of variation of u for given x and y as z varies. Comparisons can be made of other quantities regarding their variations with respect to x, y, and z and similar observations can be made. Computations were performed for $R > 100$ but no results are presented here because the grid sizes of $h = 0.04$ and larger were found not to give accurate results.

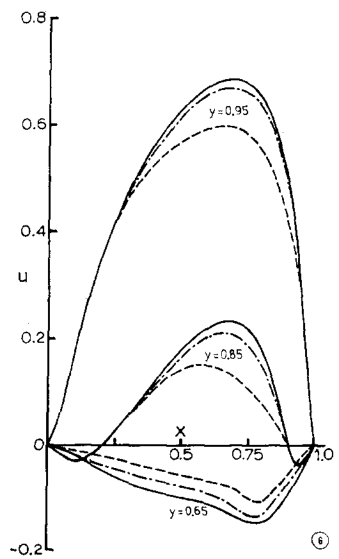

FIG. 6. The variation of u at $R = 100$ as a function of x for various values of y and z. ——, $z = 0$; - - - - -, $z = 0.3$; – – –, $z = 0.4$.

There was no difficulty in obtaining the numerical results using the finite-difference formulation given by Eqs. (11) without the use of any underrelaxation parameters but when using the finite-difference formulation given by Eqs. (8) and (10) an underrelaxation procedure had to be adopted in order to obtain convergent solutions.

Pan and Acrivos [17] considered experimentally the flow in rectangular cavities for various aspect ratios. In particular, results were obtained for the cubical cavity considered in the present paper. They state that the three-dimensional effects of the motion were important in certain regions but not in the central section corresponding to the plane $z = 0$ in the present notation, where to all appearances the flow was indeed two dimensional. Pan and Acrivos further state that at this central section excellent agreement with the numerical solution given by Burggraf [20] was obtained up to $R = 400$. Since the three-dimensional results at $z = 0$ obtained up to $R = 100$ in the present paper differ by at most 2–3 % from those obtained by solving Burggraf's two-dimensional problem, then they must also be in good agreement with the experimental results in the plane $z = 0$. No experimental results were given for any other values of z and therefore it is not possible to make further comparisons with the results of the experiments.

5. Conclusions

The purpose of this paper has been to present a study of several numerical schemes of solving the steady three-dimensional Navier–Stokes equations. The sets of finite-difference equations (10) and (11) have not previously been applied to problems in three dimensions. It has been found that the set of Eqs. (11) give rise to very stable numerical procedures for approximating the equations governing the transport of vorticity and that they can be solved quite rapidly by standard iterative procedures. In the example considered of flow inside a cubical box, results have been presented only for values of the Reynolds numbers up to 100, but the method based on Eq. (11) has been found to be stable for considerably higher Reynolds numbers. However, the question of the accuracy of the results must also be considered. The results presented up to $R = 100$ have been investigated for accuracy by varying the grid size and it was judged that smaller grid sizes would be necessary to give reliable results for $R > 100$ with a consequent increase in computer storage requirements and execution time.

It appears from the present results that the three-dimensional flow in the central plane $z = 0$ of the box is approximated quite well for Reynolds numbers up to 100 by the two-dimensional flow obtained by assuming the length of the box to be infinite in the z direction. The numerical results of Burggraf [20] appear also to be in good agreement with the experimental results of Pan and Acrivos [17] on the plane $z = 0$ for values of R up to 400. The work of De Vahl Davis and Mallinson [12] suggests that the two-dimensional model at $R = 100$ and 400 overestimates the strength of the three-dimensional motion in the plane $z = 0$. However, the results were based on the use of a grid $h = 1/15$ and this was not found to be adequate in either the two-dimensional calculations of Burggraf or those of the present investigation except for small values of the Reynolds number. The effect of reducing the grid size at a given Reynolds number is to give results in which the strength of the motion is increased. It is a conclusion of the present work that further experimentation with variation of grid size is necessary in the three-dimensional problem for Reynolds numbers much greater than 100.

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REFERENCES

1. D. N. DE G. ALLEN AND R. V. SOUTHWELL, *Quart. J. Mech. Appl. Math.* **8** (1955), 129.
2. S. C. R. DENNIS, *Quart. J. Mech. Appl. Math.* **13** (1960), 487.
3. D. B. SPALDING, *Int. J. Num. Meth. Eng.* **4** (1972), 551.
4. S. C. R. DENNIS, *in* "Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics," Lecture Notes in Physics, Vol. 19, p. 120, Springer–Verlag, Berlin/New York, 1973.
5. D. F. ROSCOE, *J. Inst. Math. Appl.* **16** (1975), 291.
6. D. F. ROSCOE, *Int. J. Num. Meth. Eng.* **10** (1976), 1299.
7. S. C. R. DENNIS AND J. D. HUDSON, *in* "Proceedings of the International Conference on Numerical Methods in Laminar and Turbulent Flow, Swansea, United Kingdom," p. 69, Pentech Press, London, 1978.
8. A. J. CHORIN, *Math. Comp.* **22** (1968), 745.
9. G. P. WILLIAMS, *J. Fluid Mech.* **37** (1969), 727.
10. K. AZIZ AND J. D. HELLUMS, *Phys. Fluids* **10** (1967), 314.
11. G. D. MALLINSON AND G. DE VAHL DAVIS, *J. Computational Phys.* **12** (1973), 435.
12. G. DE VAHL DAVIS AND G. D. MALLINSON, *Comput. & Fluids* **4** (1976), 29.
13. S. M. RICHARDSON AND A. R. H. CORNISH, *J. Fluid Mech.* **82** (1977), 309.
14. G. D. MALLINSON AND G. DE VAHL DAVIS, *J. Fluid Mech.* **83** (1977), 1.
15. R. N. COOK, Ph. D. thesis, University of Western Ontario, London, Ontario, Canada, 1977.
16. R. S. VARGA, "Matrix Iterative Analysis," Prentice–Hall, Englewood Cliffs, N. J., 1962.
17. F. PAN AND A. ACRIVOS, *J. Fluid Mech.* **28** (1967), 643.
18. D. GREENSPAN, "Lectures on the Numerical Solution of Linear, Singular and Nonlinear Differential Equations," Prentice–Hall, Englewood Cliffs, N. J., 1968.
19. A. D. GOSMAN, W. M. PUN, A. K. RUNCHAL, D. B. SPALDING, AND M. WOLFSHTEIN, "Heat and Mass Transfer in Recirculating Flows," Academic Press, New York, 1969.
20. O. R. BURGGRAF, *J. Fluid Mech.* **24** (1966), 113.